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From extended regular variation to regular variation with application in extreme value statistics

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ABSTRACT

This paper aims to provide a study of a variety of concepts involving power behavior of eventually positive functions which, falling under the umbrella of the Theory of Regular Variation and its second order refinements, are prone to application in Extreme Value Theory. To this extent, some well-known properties shall be resumed, others will be designed with the ultimate purpose of establishing a relation between regular variation and extended regular variation of second order. As a by-product, new ways of looking at some common estimators for the extreme value index, in particular the maximum likelihood estimator, will be unveiled.

1. Introduction and main result

Regular variation theory has been a prolific tool in Extreme Value Theory since it provides a suitable framework to study key features and properties of distribution functions belonging to extreme domains of attraction, with emphasis on results pertaining to consistency and asymptotic distribution characterization. Under the umbrella of Regular Variation Theory (see e.g. [2,4]), we provide in this paper a set of tools which are bound to be used together with procedures for modeling and extrapolating beyond the sampled values. We shall resume and broaden the work of Fraga Alves et al. [9], thus establishing how to transfer results within the scope of extended regular variation to the class of functions determined by plain regular variation in a second order framework. For detailed coverage on the essentials of regular variation theory (of second order) we refer to de Haan and Stadtmüller [5] and de Haan and Ferreira [4].

The analysis of extreme values rests essentially on the fundamental Fisher–Tippett theorem [8], which ultimately projects the Generalized Extreme Value distribution as a unified version of all possible non-degenerate weak limits of partial maxima of a sequence of independent random variables $\{X_n\}_{n\geqslant 1}$ with the same distribution function F. That is, assume there exist normalizing constants $a_n>0$ and $b_n\in\mathbb{R}$ such that

$$\lim_{n\to\infty} P\left\{a_n^{-1}\left(\max(X_1,\dots,X_n)-b_n\right)\leqslant x\right\}=\lim_{n\to\infty} F^n(a_n x+b_n)=G(x), \tag{1}$$

for all x, with some non-degenerate distribution function G. Then, G must be one of only three possible extreme value distributions – Fréchet, Gumbel or Weibull – while these, in turn, can be nested as a one parameter family of distributions [11]: the Generalized Extreme Value distribution,

$$G_\gamma(x):=\begin{cases}\exp(-(1+\gamma x)^{-1/\gamma}), & 1+\gamma x>0 \quad \text{if } \gamma\neq 0,\\ \exp(-\exp(-x)), & x\in\mathbb{R} \qquad\quad\ \text{if } \gamma=0.\end{cases} \tag{2}$$

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We say that the distribution function F of the independent random variables $X_1, X_2, \dots, X_n, \dots$ is in the domain of attraction of G_γ if (1) holds with $G = G_\gamma$. The following *extended regular variation* property [3] is a well-known necessary and sufficient condition for F to belong to an extreme domain of attraction:

$$\lim_{t\to\infty} \frac{U(tx)-U(t)}{a(t)} = \begin{cases} \frac{x^\gamma-1}{\gamma}, & \gamma \neq 0, \\ \log x, & \gamma = 0, \end{cases} \tag{3}$$

for every $x > 0$ and some positive measurable function a, with U standing for a quantile type function pertaining to F defined by the generalized inverse

$$U(t) := \left(\frac{1}{1-F}\right)^{\leftarrow}(t) = \inf\left\{x\colon\ F(x) \geqslant 1 - \frac{1}{t}\right\}.$$

On account of simplicity, we assume throughout that $U(\infty) := \lim_{t\to\infty} U(t) > 0$.

Apart from the first order condition (3), we shall need a second order condition, specifying the inherent rate of the convergence. Hence, we shall assume the existence of a function A, not changing in sign and tending to zero as $t \to \infty$ such that

$$\lim_{t\to\infty} \frac{\frac{U(tx)-U(t)}{a(t)} - \frac{x^\gamma-1}{\gamma}}{A(t)} = H_{\gamma,\rho}(x) := \frac{1}{\rho}\left(\frac{x^{\gamma+\rho}-1}{\gamma+\rho} - \frac{x^\gamma-1}{\gamma}\right), \tag{4}$$

for all $x > 0$, where $\rho \leqslant 0$ is the second order parameter governing the speed of convergence in (3). We then say that the function U is of *second order extended regular variation* (notation: $U \in 2ERV(\gamma,\rho)$). In case $\rho = 0$ and/or $\gamma + \rho = 0$, note that the function $(x^a - 1)/a$ defined for all $x > 0$, $a \in \mathbb{R}$ reads as $\log x$ for $a = 0$. We also remark that $\lim_{t\to\infty} A(tx)/A(t) = x^\rho$, for every $x > 0$, i.e., $|A|$ is of regular variation at infinity with index ρ (cf. [5]), hence the notation $|A| \in RV_\rho$.

In connection with the class of functions $2ERV$, determined by (4), the following theorem is the prominent result of this paper. It lengthens the result in Theorem 3.1 of [9] for general γ. The proof however shall be postponed to Section 3.

Theorem 1. *Assume $U \in 2ERV(\gamma,\rho)$ with $\gamma \neq \rho$. Then*

$$\lim_{t\to\infty} \frac{\frac{U(tx)}{U(t)} - x^{\gamma_+}}{\widetilde{A}(t)} = \widetilde{H}_{\gamma,\rho}(x), \tag{5}$$

for all $x > 0$, where $\gamma_+ := 0 \vee \gamma$,

$$\widetilde{H}_{\gamma,\rho}(x) := \begin{cases} x^\gamma \frac{x^\rho-1}{\rho}, & 0 \leqslant -\rho < \gamma \text{ or } (0 < \gamma < -\rho \text{ and } l = 0), \\ x^{\gamma_+} \frac{x^{-|\gamma|}-1}{-|\gamma|}, & (0 < \gamma < -\rho \text{ and } l \neq 0) \text{ or } \gamma = -\rho > 0 \text{ or } \gamma \leqslant 0, \end{cases}$$

$$\widetilde{A}(t) = \begin{cases} \frac{\gamma}{\gamma+\rho} A(t), & 0 \leqslant -\rho < \gamma \text{ or } (0 < \gamma < -\rho \text{ and } l = 0), \\ \frac{a(t)}{U(t)} - \gamma_+, & (0 < \gamma < -\rho \text{ and } l \neq 0) \text{ or } \gamma = -\rho > 0 \text{ or } \gamma \leqslant 0 \end{cases} \tag{6}$$

and $l := \lim_{t\to\infty}(U(t) - a(t)/\gamma)$. Furthermore, $|\widetilde{A}(t)| \in RV_{\widetilde{\rho}}$ with

$$\widetilde{\rho} = \begin{cases} \rho, & 0 \leqslant -\rho < \gamma \text{ or } (0 < \gamma < -\rho \text{ and } l = 0), \\ -|\gamma|, & (0 < \gamma < -\rho \text{ and } l \neq 0) \text{ or } \gamma = -\rho > 0 \text{ or } \gamma \leqslant 0. \end{cases}$$

The main Theorem 1 and the important consequences attached below in this paper offer a unified treatment of the asymptotic properties carried out for most estimators of extreme value characteristics we find coexisting in the literature.

The remainder of this paper is laid out as follows. Section 2 contains ancillary results to the proof of the main result which is developed in Section 3, as well as to the subsequent results introduced in Section 4 involving progression from extended regular variation to regular variation of second order, with grasp to a third order framework. These conditions for linkage between plain and extended RV will enable further insight into asymptotic behavior of some widely used estimators for the extreme value index $\gamma \in \mathbb{R}$ (featuring in (2)), often regarded as a gauge of tail heaviness of the underlying distribution function F. The latter also applies to other extreme characteristics such as, e.g., high quantile estimation. Because the asymptotic bias of such estimators is in fact paired at the core with the auxiliary function $\widetilde{A}$ displayed in (6), the order of this dominant component should be kept under scrutiny, at the risk of bias disruption by inflicting on the data a shift by a constant $l \in \mathbb{R}$, say. This is the instant message carried by the main Theorem 1, with particular relevance for the heavy-tailed case, i.e., in case the unknown parent distribution function F is associated with a positive extreme value index γ. Obviously, bias reduction could also be attained under the same scheme. In Section 5, an example of application is provided where a simplified proof for asymptotic normality of the maximum likelihood estimator (cf. [7]) hinges on the collateral results encompassed in Section 4 while focusing on heavy tails. The maximum likelihood estimator we refer to is a shift-invariant estimator for the shape parameter γ in the sense that it sustains any possible shift in location of the data.

2. Auxiliary results

In this section we state without proof some results just about to used in Section 3 for the proof of the main theorem, and onwards to verify results enclosed in Section 4. Despite these first lemmas have been labeled as mere auxiliary, they shed some clarity on whether we can make progression from (second order) RV to (second order) extended regular variation without narrowing much the class of distribution functions initially considered, i.e., those distributions lying in some extreme domain of attraction such that (4) is also satisfied.

It is worthwhile to mention that the following results hold with any measurable (eventually) positive function U.

Lemma 2. *(See [4, Theorem B.2.2 and Corollary B.2.13].) Suppose the function U is such that* (3) *holds with some $\gamma \in \mathbb{R}$. Then, the auxiliary function a is of regular variation with index γ, i.e., $a \in RV_\gamma$ and*

(i) *If $\gamma > 0$, then $\lim_{t\to\infty} U(t) = \infty$, $\lim_{t\to\infty} a(t)/U(t) = \gamma$ and U belongs to RV_γ.*
(ii) *If $\gamma < 0$, then $U(\infty) < \infty$, $\lim_{t\to\infty} a(t)/(U(\infty) - U(t)) = -\gamma$ and $U(\infty) - U(t)$ is in RV_γ.*
(iii) *if $\gamma = 0$, then U is of slow variation in the sense that $U \in RV_0$, i.e., $\lim_{t\to\infty} U(tx)/U(t) = 1$, for every $x > 0$, and $a(t) = o(U(t))$, as $t \to \infty$.*

Lemma 3. *(See [4, Theorem B.3.6].) Suppose $U \in 2ERV(\gamma, \rho)$ as introduced in* (4)*. Then,*

(i) *in case $\rho = 0$ and $\gamma > 0$,*

$$\lim_{t\to\infty} \frac{(tx)^{-\gamma} f(tx) - t^{-\gamma} f(t)}{t^{-\gamma} f(t) A(t)} = \log x, \tag{7}$$

for $x > 0$, with $f(t) := U(t)$;
(ii) *in case $\rho = 0$ and $\gamma < 0$,* (7) *holds with $f(t) := U(\infty) - U(t)$;*
(iii) *in case $\rho < 0$, $\lim_{t\to\infty} t^{-\gamma} a(t) = c$ exists, $c \in (0, \infty)$, and*

$$\lim_{t\to\infty} \frac{f(tx) - f(t)}{a(t)A(t)/\rho} = \frac{x^{\gamma+\rho} - 1}{\gamma + \rho}, \tag{8}$$

for $x > 0$, with $f(t) := U(t) - c(t^\gamma - 1)/\gamma$.

Conversely, any of the three properties above implies that $U \in 2ERV(\gamma, \rho)$ with the indicated auxiliary function A.

If we require $\gamma + \rho \neq 0$ in Lemma 3(iii) and take $A(t)$ such that $\lim_{t\to\infty} t^{-\rho} A(t) = c_2^\star \neq 0$, then (8) (and thus relation (4) under the same restrictions) is equivalent to

$$U(t) = c_0 + c\frac{t^\gamma - 1}{\gamma} + c_2 \frac{t^{\gamma+\rho}}{\gamma+\rho} + o\big(t^{\gamma+\rho}\big) = c\frac{t^\gamma}{\gamma}\left(1 - \frac{\gamma}{\gamma+\rho} c_2 t^\rho + (\gamma c_0 - c)t^{-\gamma} + o\big(t^\rho\big)\right), \tag{9}$$

as $t \to \infty$, with $c_0, c_2 \neq 0$. Provided $\gamma > 0$ and $\rho < 0$, the expansion above transpires that the rate of convergence of $U(tx)/U(t)$ to the limit x^γ might be hampered or improved by means of a shift on the function U. Whenever $\gamma + \rho < 0$, it is easily seen from the last equality in (9) that a shift by a constant in the function U may alter the power of second order, given the important role played by c_0 in this case. The main Theorem 1 also exploits the eventual connection between a shift by $l = c_0 - c/\gamma \neq 0$ and a possible change in the inherent rate of convergence to the regular variation of U.

The next lemma aims to describe how fast the auxiliary function $a \in RV_\gamma$ (from (3) and by Lemma 2, respectively) approaches the simple regularly varying function ct^γ in the first place. The second part of the lemma is only to be used in the proof of Theorem 15 with respect to third order conditions.

Lemma 4. *Assume $U \in 2ERV(\gamma, \rho)$ subject to $\rho < 0$. Then*

(i) *the following limit holds*:

$$\lim_{t\to\infty} \frac{\frac{ct^\gamma}{a(t)} - 1}{A(t)} = -\frac{1}{\rho}; \tag{10}$$

(ii) *if $0 < \gamma < -\rho$, then*

$$\lim_{t\to\infty} \frac{(\frac{a(t)}{U(t)} - \gamma)^2}{A(t)} = \begin{cases} 0, & 0 < -\frac{\rho}{2} < \gamma < -\rho \text{ or } (0 < \gamma \leqslant -\frac{\rho}{2} \text{ and } l = 0), \\ \pm\infty, & 0 < \gamma < -\frac{\rho}{2} \text{ and } l \neq 0, \end{cases} \tag{11}$$

holds with

$$l := \lim_{t\to\infty}\left(U(t) - \frac{a(t)}{\gamma}\right) \in \mathbb{R}.$$

Proof. (i) From the seminal work of de Haan and Stadtmüller [5], it is well known that if the second order condition (4) holds, then

$$\lim_{t\to\infty} \frac{\frac{a(tx)}{a(t)} - x^\gamma}{A(t)} = x^\gamma \frac{x^\rho - 1}{\rho}, \tag{12}$$

for all $x > 0$ (see also [4, Theorem B.3.1]). Straightforward calculations yield (12) equivalent to

$$\lim_{t\to\infty} \frac{c^{-1}(tx)^{-\gamma}a(tx) - c^{-1}t^{-\gamma}a(t)}{c^{-1}t^{-\gamma}A(t)a(t)} = \frac{x^\rho - 1}{\rho}, \quad x > 0.$$

Recall from Lemma 3(iii) that $\lim_{t\to\infty} c^{-1}t^{-\gamma}a(t) = 1$. Hence, owing to Lemma 2(ii) with U conveniently replaced by $c^{-1}t^{-\gamma}a(t)$ for the purpose, with corresponding auxiliary function, we get

$$\lim_{t\to\infty} \frac{1 - c^{-1}t^{-\gamma}a(t)}{A(t)c^{-1}t^{-\gamma}a(t)} = -\frac{1}{\rho},$$

which immediately implies the result.

(ii) The given statement bears on the limit relation (8) since its left-hand side translates into

$$\frac{U(tx) - \frac{c(tx)^\gamma}{\gamma} - (U(t) - \frac{ct^\gamma}{\gamma})}{a(t)A(t)/\rho} - \frac{U(tx) - \frac{a(tx)}{\gamma} - (U(t) - \frac{a(t)}{\gamma})}{a(t)A(t)/\rho} = \frac{1}{\gamma}\left(\frac{a(tx)}{a(t)}\frac{A(tx)}{A(t)}\frac{1 - c(tx)^\gamma/a(tx)}{A(tx)/\rho} - \frac{1 - ct^\gamma/a(t)}{A(t)/\rho}\right).$$

Keeping in mind that $a \in RV_\gamma$ and $|A| \in RV_\rho$, we may consider the limit by applying (10) to the second member of the equality above so that (8) ultimately rephrases as

$$\lim_{t\to\infty} \frac{U(tx) - \frac{a(tx)}{\gamma} - (U(t) - \frac{a(t)}{\gamma})}{-a(t)A(t)/\gamma} = \frac{x^{\gamma+\rho} - 1}{\gamma + \rho}, \quad x > 0. \tag{13}$$

Similarly as before, assuming $\gamma + \rho < 0$, Lemma 2(ii) now ensures $l := \lim_{t\to\infty}(U(t) - a(t)/\gamma) \in \mathbb{R}$ and moreover, the following limit to hold:

$$\lim_{t\to\infty} \frac{l - (U(t) - \frac{a(t)}{\gamma})}{a(t)A(t)} = \frac{1}{\gamma(\gamma + \rho)}. \tag{14}$$

The latter implies that the left-hand side of (14),

$$\frac{1}{A(t)}\left(\frac{l}{a(t)} - \left(\frac{U(t)}{a(t)} - \frac{1}{\gamma}\right)\right),$$

multiplied by $U(t)/a(t) - \gamma^{-1}$ (which is $o(1)$ by Lemma 2(i)) vanishes for large enough t, i.e.,

$$\left(U(t) - \frac{a(t)}{\gamma}\right)\frac{l}{(a(t))^2A(t)} - \left(\frac{U(t)}{a(t)} - \frac{1}{\gamma}\right)^2 \frac{1}{A(t)} \underset{t\to\infty}{\longrightarrow} 0. \tag{15}$$

Therefore, if $l = 0$, the limit in (15) immediately yields $(U(t)/a(t) - \gamma^{-1})^2 = o(A(t))$. If $l \neq 0$, attending to the fact that $(a(t))^2|A(t)| \in RV_{2\gamma+\rho}$, we then have to take into account that the first term in (15) may not be finite, eventually:

$$\left(U(t) - \frac{a(t)}{\gamma}\right)\frac{l}{(a(t))^2A(t)} \underset{t\to\infty}{\longrightarrow} \begin{cases} 0, & \text{if } 2\gamma + \rho > 0, \\ \pm\infty, & \text{if } 2\gamma + \rho < 0 \end{cases} \quad (\gamma > 0).$$

Hence the result. □

For ease of exposition, the case $2\gamma + \rho = 0$ and $l \neq 0$ was obviated in the lemma above. However, a great variety of heavy-tailed distributions detaining second order parameter $\rho = -2\gamma$ often satisfy

$$U(t) = l + \frac{c}{\gamma}t^\gamma + c_2t^{-\gamma} + o\big(t^{-\gamma}\big) \quad (t \to \infty)$$

with $l, c, c_2 \neq 0$, thus assigning the value $l^2/(2c^2c_2) \neq 0$ to the limit in (11). For a separate treatment of this type of models, see Section 4 of [9].

3. Proof of the main theorem

Before proceeding with the proof of the main Theorem 1, let us emphasize the following:

Remark 5. Even in the heavy tailed case, i.e., in case $\gamma > 0$, it is not always true that (5) implies (4). Lemma 3(iii) accounts for this, simply make $f(t) = U(t) - c(t^\gamma - 1)/\gamma = 0$ and notice that

$$\frac{U(tx)}{U(t)} - x^\gamma = \frac{\gamma}{1-t^\gamma} x^\gamma \frac{x^{-\gamma}-1}{-\gamma}, \quad x > 0.$$

Proof. The bulk of the proof is supported on Lemma 3. This is the main reason why we shall consider the cases $\rho < 0$ and $\rho = 0$ separately.

Case $\rho < 0$: According to (iii) of Lemma 3, we may write $U(t)$ as

$$U(t) = f(t) + c\frac{t^\gamma - 1}{\gamma},$$

with f satisfying (8) and $c > 0$ such that $\lim_{t\to\infty} t^{-\gamma} a(t) = c$. Hence,

$$\frac{U(tx)}{U(t)} - x^{\gamma_+} = \frac{f(tx) - f(t) - (x^{\gamma_+} - 1)(f(t) - \frac{c}{\gamma}) + \frac{ct^\gamma}{\gamma}(x^\gamma - x^{\gamma_+})}{f(t) + \frac{ct^\gamma}{\gamma} - \frac{c}{\gamma}}. \tag{16}$$

Throughout this proof we take $a_f(t) := a(t)A(t)/\rho$.

Regarding $\gamma + \rho \geqslant 0$ while giving heed to (8), Lemma 2(i) then entails $a_f(t)/f(t) \to \gamma + \rho$, as $t \to \infty$. Using (16), we can primarily write

$$\frac{U(tx)}{U(t)} - x^\gamma = \frac{f(t)c^{-1}t^{-\gamma}}{\frac{f(t)}{ct^\gamma} + \frac{1}{\gamma} - \frac{t^{-\gamma}}{\gamma}} \left(\frac{f(tx) - f(t)}{a_f(t)} \frac{a_f(t)}{f(t)} - \left(x^\gamma - 1\right)\left(1 - \frac{1}{c^{-1}\gamma f(t)}\right)\right).$$

Since $c^{-1}t^{-\gamma} a_f(t) = \rho^{-1}A(t)(1 + o(1))$, as $t \to \infty$, we thus get for $\gamma + \rho > 0$,

$$\frac{U(tx)}{U(t)} - x^\gamma = \frac{\gamma}{\gamma+\rho} \frac{a_f(t)}{a(t)} \frac{a(t)}{ct^\gamma} x^\gamma \left(x^\rho - 1\right)\left(1 + o(1)\right) + o\left(t^\rho\right) = \frac{\gamma}{\gamma+\rho} A(t) x^\gamma \frac{x^\rho - 1}{\rho} + o\left(A(t)\right),$$

whereas for $\gamma = -\rho$ we may only write for now that

$$\frac{U(tx)}{U(t)} - x^\gamma = \left(\frac{f(tx) - f(t)}{a_f(t)} \frac{\gamma a_f(t)}{ct^\gamma} + \frac{x^\gamma - 1}{\gamma} \gamma t^{-\gamma} \left(1 - \gamma c^{-1} f(t)\right)\right)\left(1 + o(1)\right). \tag{17}$$

We can tidy up the right-hand side of (17) by recalling from Lemma 3(iii) that $a(t) \sim ct^\gamma$, as $t \to \infty$, and Lemma 2(i) on that $\lim_{t\to\infty} a(t)/U(t) = \gamma$, since these yield the following expansions:

$$\gamma t^{-\gamma}\left(1 - \gamma c^{-1} f(t)\right) = -\gamma^2 c^{-1} t^{-\gamma} \left(U(t) - \frac{a(t)}{\gamma}\right)\left(1 + o(1)\right) = \left(\frac{a(t)}{U(t)} - \gamma\right)\left(1 + o(1)\right), \tag{18}$$

which are valid for any $\gamma > 0$ and $\rho < 0$. Added the fact elapsing from relation (13) with $\gamma + \rho = 0$, through application of Lemma 2(iii) that

$$\lim_{t\to\infty} \frac{a(t)/U(t) - \gamma}{A(t)} = \pm\infty,$$

the first term on the right-hand side of (17) can be discarded. The result thus follows.

In case $\gamma + \rho < 0$, once Lemma 2(ii) upon (8) ascertains the existence of $f(\infty) := \lim_{t\to\infty} f(t)$ we find useful to write the following equality related to (16):

$$\begin{aligned} \frac{U(tx)}{U(t)} - x^{\gamma_+} = {} & \left\{ t^{-\gamma}\left(f(t) - \frac{c}{\gamma}\right) + \frac{c}{\gamma}\right\}^{-1} \left\{ \frac{f(tx) - f(t)}{a_f(t)} \frac{a_f(t)}{f(\infty) - f(t)} t^{-\gamma}\left(f(\infty) - f(t)\right) \right. \\ & \left. + \left(x^{\gamma_+} - 1\right) t^{-\gamma}\left(f(\infty) - f(t)\right) - \left(f(\infty) - \frac{1}{\gamma}\right)\left(x^{\gamma_+} - 1\right)t^{-\gamma} + \frac{c}{\gamma}\left(x^\gamma - x^{\gamma_+}\right)\right\}. \end{aligned} \tag{19}$$

Now, Lemma 2(ii) furthermore ensures $a_f(t)/(f(\infty) - f(t)) \to -(\gamma + \rho)$, as $t \to \infty$, and thus $t^{-\gamma}(f(\infty) - f(t)) = O(A(t))$. In this sequence, we get from (19) that:

If $\gamma > 0$ (hence $\rho < -\gamma < 0$), then (19) admits the representation

$$\frac{U(tx)}{U(t)} - x^\gamma = \frac{A(t)}{\rho}\frac{\gamma}{\gamma+\rho}x^\gamma\big(x^\rho - 1\big) + t^{-\gamma}\big(1 - \gamma c^{-1} f(\infty)\big)\big(x^\gamma - 1\big) + o\big(t^{-\gamma}\big) + o\big(A(t)\big). \tag{20}$$

Note that $-\gamma c^{-1}\lim_{t\to\infty}(U(t) - a(t)/\gamma) = 1 - \gamma c^{-1} f(\infty)$. Denoting by l the limit in the latter expression, we obtain

$$\frac{U(tx)}{U(t)} - x^\gamma = \begin{cases} -\gamma^2\frac{t^{-\gamma}}{c}(U(t) - \frac{a(t)}{\gamma})x^\gamma\frac{x^{-\gamma}-1}{-\gamma} + o(t^{-\gamma}(U(t) - \frac{a(t)}{\gamma})), & l \neq 0, \\ \frac{\gamma}{\gamma+\rho}A(t)x^\gamma\frac{x^\rho-1}{\rho} + o(A(t)), & l = 0, \end{cases} \tag{21}$$

which, supported on the second equality in (18) together with the statement built on Lemma 2(i),

$$\frac{a(t)}{U(t)} - \gamma = -\gamma\frac{a(t)}{U(t)}\left(\frac{U(t)}{a(t)} - \frac{1}{\gamma}\right) = -\gamma^2\left(\frac{U(t)}{a(t)} - \frac{1}{\gamma}\right)\big(1 + o(1)\big) \quad (t \to \infty)$$

leads to the result almost immediately.

If $\gamma < 0$ (which implies $\gamma + \rho < 0$), then (19) determines the equality:

$$\frac{U(tx)}{U(t)} - 1 = \frac{ct^\gamma}{f(t) - \frac{c}{\gamma} + \frac{c}{\gamma}t^\gamma}\left\{\frac{x^\gamma - 1}{\gamma} + \frac{f(tx) - f(t)}{a_f(t)}\frac{a_f(t)}{f(\infty) - f(t)}\frac{f(\infty) - f(t)}{ct^\gamma}\right\}. \tag{22}$$

Again, Lemma 2(ii) upon (8) ascertains $f(\infty) - f(t) = o(1)$. Hence, we use Taylor's expansion upon the first factor in (22):

$$ct^\gamma\left(\frac{1}{f(\infty) - c/\gamma - (f(\infty) - f(t)) + c/\gamma t^\gamma}\right) = \frac{ct^\gamma}{f(\infty) - c/\gamma}\left(1 - \frac{ct^\gamma}{f(\infty) - c/\gamma}\right)\big(1 + o(1)\big).$$

When we apply Lemma 3(iii) to the second factor in (22), while noting that it entails the limit of the function f (defined therein) exists and verifies $U(\infty) = f(\infty) - c/\gamma \in (0, \infty)$, we finally obtain

$$\begin{aligned}\frac{U(tx)}{U(t)} - 1 &= \frac{ct^\gamma}{f(\infty) - \frac{c}{\gamma}}\left(\frac{x^\gamma - 1}{\gamma} - \big(x^{\gamma+\rho} - 1\big)\frac{f(\infty) - f(t)}{ct^\gamma}\right)\left(1 - \frac{ct^\gamma}{f(\infty) - \frac{c}{\gamma}}\right)\big(1 + o(1)\big) \\ &= \frac{ct^\gamma}{U(\infty)}\frac{x^\gamma - 1}{\gamma} + o\big(t^\gamma\big) \quad (t \to \infty).\end{aligned} \tag{23}$$

The latter thus allow to devise the limit relation (recall $\lim_{t\to\infty} t^{-\gamma}a(t) = c$):

$$\lim_{t\to\infty}\frac{\frac{U(tx)}{U(t)} - 1}{a(t)/U(t)} = \frac{x^\gamma - 1}{\gamma}, \quad x > 0. \tag{24}$$

If $\gamma = 0$ and $\rho < 0$, it suffices to interpret $(x^\gamma - 1)/\gamma$ in the limiting sense as $\log x$ so that, for large enough t, (19) amounts to

$$\frac{U(tx)}{U(t)} - 1 = \left(\log t + \frac{f(\infty)}{c}\right)^{-1}\left(\frac{f(tx) - f(t)}{a_f(t)}\frac{a_f(t)}{c} + \log x\right)\big(1 + o(1)\big).$$

Whence

$$\frac{U(tx)}{U(t)} - 1 = \left(\log t + \frac{f(\infty)}{c}\right)^{-1}\left(\frac{x^\rho - 1}{\rho}\frac{A(t)}{\rho} + \log x\right)\big(1 + o(1)\big).$$

The precise result follows by taking into account that $|A(t)\log t| \in RV_\rho$ together with Lemma 3 as it ascertains $a(t)/U(t) \sim (\log t)^{-1}$, $t \to \infty$, thus yielding

$$\frac{U(tx)}{U(t)} - 1 = \frac{\log x}{\log t} + o\big((\log t)^{-1}\big) = \frac{a(t)}{U(t)}\log x + o\left(\frac{a(t)}{U(t)}\right). \tag{25}$$

Case $\rho = 0$: If $\gamma > 0$, by straightforward calculations we can write (7) in the form

$$\lim_{t\to\infty}\frac{\frac{U(tx)}{U(t)} - x^\gamma}{A(t)} = x^\gamma\log x,$$

for all $x > 0$, whereas for $\gamma < 0$, condition (7) yields

$$\lim_{t\to\infty}\frac{\frac{U(\infty)-U(tx)}{U(\infty)-U(t)} - x^\gamma}{A(t)} = x^\gamma\log x, \quad x > 0,$$

which entails in turn that

$$\left(\frac{U(tx)}{U(t)}-1\right)\frac{U(t)}{a(t)}=\frac{U(\infty)-U(t)}{a(t)}\bigl(1-x^{\gamma}-A(t)x^{\gamma}\log x+o\bigl(A(t)\bigr)\bigr)\quad(t\to\infty).$$

The precise result follows from the above in conjunction with the statement from Lemma 2(ii) that guarantees the existence of $\lim_{t\to\infty}(U(\infty)-U(t))/a(t)=-1/\gamma$. $\square$

4. Collateral results

As announced in the introduction, the results comprising this section allow to foresee useful ramifications of Theorem 1. Although the present statements involving progression from 2ERV to second order RV are akin to the main Theorem 1, the result in Theorem 15 below regards a suitable yet reasonable third order refinement with impending application in the context of extreme value analysis (cf. Section 5).

Proposition 6. *Suppose* (4) *holds with* $\gamma\neq\rho$. *Then,*

$$\lim_{t\to\infty}\frac{\frac{U(t)}{U(tx)}-x^{-\gamma_+}}{\widetilde{A}(t)}=K_{\gamma,\widetilde{\rho}}(x):=-x^{-\gamma_+}\frac{x^{\widetilde{\rho}}-1}{\widetilde{\rho}},\tag{26}$$

for all $x>0$, *where*

$$\widetilde{A}(t)=\begin{cases}\frac{\gamma}{\gamma+\rho}A(t), & 0\leqslant-\rho<\gamma \text{ or } (0<\gamma<-\rho \text{ and } l=0),\\ \frac{a(t)}{U(t)}-\gamma_+, & (0<\gamma<-\rho \text{ and } l\neq 0) \text{ or } \gamma=-\rho \text{ or } \gamma\leqslant 0,\end{cases}$$

$|\widetilde{A}(t)|\in RV_{\widetilde{\rho}}$, *with* $l:=\lim_{t\to\infty}(U(t)-a(t)/\gamma)$ *and*

$$\widetilde{\rho}=\begin{cases}\rho, & 0\leqslant-\rho<\gamma \text{ or } (0<\gamma<-\rho \text{ and } l=0),\\ -|\gamma|, & (0<\gamma<-\rho \text{ and } l\neq 0) \text{ or } \gamma=-\rho \text{ or } \gamma\leqslant 0.\end{cases}\tag{27}$$

Proof. Assembling (5) introduced in Theorem 1 with the fact that, for general γ,

$$\frac{U(tx)}{U(t)}-x^{\gamma_+}=-x^{\gamma_+}\frac{U(tx)}{U(t)}\left(\frac{U(t)}{U(tx)}-x^{-\gamma_+}\right)=-x^{2\gamma_+}\left(\frac{U(t)}{U(tx)}-x^{-\gamma_+}\right)\bigl(1+o(1)\bigr),\tag{28}$$

as $t\to\infty$, the result follows with $K_{\gamma,\rho}(x)=-x^{-2\gamma_+}\widetilde{H}_{\gamma,\rho}(x)$ by straightforward calculations. $\square$

Corollary 7. *Under the assumptions of Proposition* 6,

$$\lim_{t\to\infty}\frac{\log U(tx)-\log U(t)-\gamma_+\log x}{\widetilde{A}(t)}=\frac{x^{\widetilde{\rho}}-1}{\widetilde{\rho}}=:\widetilde{K}_{\gamma,\widetilde{\rho}}(x),\tag{29}$$

for every $x>0$.

Proof. The result follows immediately from (26). $\square$

It is noteworthy at this point that the limit relation (29) when pertaining to $\gamma\leqslant 0$ yields a somewhat more complicated situation since it resembles a first order condition. For $\gamma>0$, it already determines the usual second order condition for heavy tails. As a matter of fact, if we embed Lemma 2 with respect to $\gamma>0$ in the representation below, which follows from Theorem 1,

$$\frac{U(t)}{a(t)}\bigl(\log U(tx)-\log U(t)\bigr)=\log x+\left(\left(\frac{U(t)}{a(t)}-\frac{1}{\gamma}\right)\gamma\log x+x^{-\gamma}\left(\frac{U(tx)}{U(t)}-x^{\gamma}\right)\right)\bigl(1+o(1)\bigr),$$

as $t\to\infty$, we make the last term negligible. Hence, assembling with (29) subject to $\gamma\leqslant 0$, we get the extended regular variation of $\log U$ for general γ:

$$\lim_{t\to\infty}\frac{\log U(tx)-\log U(t)}{a(t)/U(t)}=\frac{x^{\gamma_-}-1}{\gamma_-}\quad(x>0),\ \text{ where } \gamma_-:=\gamma\wedge 0.\tag{30}$$

There are also accompanying uniform bounds in the sense of the important uniform inequalities provided by Drees [6] (see also de Haan and Ferreira [4] and references therein):

Proposition 8. *If* (5) *holds for some* $\gamma \in \mathbb{R}$*, then for each* $\varepsilon > 0$ *there exits a* $t_0 = t_0(\varepsilon)$ *such that for* $t \geqslant t_0$ *and* $x \geqslant 1$*,*

$$(1-\varepsilon)x^{-\gamma_+-\varepsilon} < \frac{U(t)}{U(tx)} < (1+\varepsilon)x^{-\gamma_++\varepsilon} \tag{31}$$

and

$$\left|\frac{\frac{U(t)}{U(tx)} - x^{-\gamma_+}}{\widetilde{A}(t)} - K_{\gamma,\widetilde{\rho}}(x)\right| \leqslant \varepsilon x^{-\gamma_+-\widetilde{\rho}+\varepsilon}. \tag{32}$$

The next proposition encompasses a second order relation restrained to the case $\rho < 0$. The corresponding result for $\rho = 0$ albeit under $\gamma \neq 0$ is provided in Lemma 11 displayed afterwards.

Proposition 9. *Let* $U \in 2ERV(\gamma,\rho)$ *and assume* $U(\infty) > 0$*. Then, restricting to* $\rho < 0$ *and* $\gamma \neq \rho$*, the following holds with* $l \in \mathbb{R}$ *defined as* $l := \lim_{t\to\infty}(U(t) - a(t)/\gamma)$ *in case* $\gamma + \rho < 0$*:*

$$\lim_{t\to\infty} \frac{\frac{(U(t))^2}{a^\star(t)}\left(-\frac{1}{U(tx)} + \frac{1}{U(t)}\right) - \frac{x^{-|\gamma|}-1}{-|\gamma|}}{A^\star(t)} = K^\star_{\gamma,\rho}(x) := \begin{cases} \frac{x^{-|\gamma|+\rho^\star}-1}{-|\gamma|+\rho^\star}, & \gamma \neq 0, \\ \frac{(\log x)^2}{2}, & \rho < \gamma = 0, \end{cases} \tag{33}$$

for all $x > 0$*, where*

$$a^\star(t) = \begin{cases} a(t)(1 - A^\star(t)), & \gamma \neq 0, \\ a(t), & \gamma = 0 \end{cases}$$

and

$$A^\star(t) = \begin{cases} \frac{\gamma_+-\rho}{-\rho(\gamma_++\rho)} A(t), & |\gamma| > -\rho \text{ or } (0 < \gamma < -\rho \text{ and } l = 0), \\ \frac{2}{|\gamma|}\left(\frac{a(t)}{U(t)} - \gamma_+\right), & (0 < \gamma < -\rho \text{ and } l \neq 0) \text{ or } \gamma = -\rho \text{ or } \rho < \gamma < 0, \\ -2\frac{a(t)}{U(t)}, & \rho < \gamma = 0. \end{cases}$$

Then $|A^\star(t)| \in RV_{\rho^\star}$ *with*

$$\rho^\star = \begin{cases} \rho, & |\gamma| > -\rho \text{ or } (0 < \gamma < -\rho \text{ and } l = 0), \\ -|\gamma|, & (0 < \gamma < -\rho \text{ and } l \neq 0) \text{ or } \gamma = -\rho \text{ or } \rho < \gamma \leqslant 0. \end{cases}$$

Proof. We find convenient to write

$$\frac{(U(t))^2}{a(t)}\left(-\frac{1}{U(tx)} + \frac{1}{U(t)}\right) = \frac{U(t)}{a(t)}\left(1 - \frac{U(t)}{U(tx)}\right) = \frac{U(t)}{U(tx)}\frac{U(tx)-U(t)}{a(t)}. \tag{34}$$

Next, we find a second order expansion for the two factors in the right-hand side of (34) separately. By the one hand, we have Proposition 6 ascertaining the expansion

$$\frac{U(t)}{U(tx)} = x^{-\gamma_+} + \widetilde{A}(t)K_{\gamma,\rho}(x)\big(1+o(1)\big) \quad (t\to\infty); \tag{35}$$

by the other hand, relying on (iii) of Lemma 3, we get from the second factor in (34) the contribution

$$\frac{U(tx)-U(t)}{a(t)} = \left(\frac{A(t)}{\rho}\frac{x^{\gamma+\rho}-1}{\gamma+\rho} + \frac{ct^\gamma}{a(t)}\frac{x^\gamma-1}{\gamma}\right)\big(1+o(1)\big).$$

Using (i) of Lemma 4, we thus obtain that indeed

$$\frac{U(t)}{U(tx)}\frac{U(tx)-U(t)}{a(t)} = \frac{x^\gamma-1}{\gamma} + \frac{A(t)}{\rho}H_{\gamma,\rho}(x) + o\big(A(t)\big) \quad (t\to\infty). \tag{36}$$

In view of (34), our purpose will be served by gathering (35) and (36) as follows:

$$\frac{U(t)}{U(tx)}\frac{U(tx)-U(t)}{a(t)} = \left(x^{-\gamma_+}\left(\frac{x^\gamma-1}{\gamma} + \frac{A(t)}{\rho}H_{\gamma,\rho}(x)\right) + \widetilde{A}(t)\frac{x^\gamma-1}{\gamma}K_{\gamma,\rho}(x)\right)\big(1+o(1)\big). \tag{37}$$

Note that, depending on the relative magnitude of γ and ρ, this function $\widetilde{A}$ is sometimes proportional to the function A, sometimes amounts to $a(t)/(U(t) - \gamma_+)$ multiplied by a constant also (see Proposition 6). Hence, rather than try to present an intermittent proof for general γ, let us focus on the cases $\gamma > 0$, $\gamma < 0$ and $\gamma = 0$ with this specific order.

Case $\gamma > 0$: Apart from the $o(\cdot)$ terms, the right-hand side of (37) can be primarily expressed as

$$\left(\frac{x^{-\gamma}-1}{-\gamma}+\frac{A(t)}{\rho}\left(\frac{x^{\gamma+\rho}-1}{\gamma+\rho}-\frac{x^{-\gamma}-1}{-\gamma}\right)+\widetilde{A}(t)\frac{x^{\gamma}-1}{\gamma}K_{\gamma,\rho}(x)\right)\big(1+o(1)\big). \tag{38}$$

If ($0<\gamma<-\rho$ and $l\neq 0$) or $\gamma=-\rho$, Proposition 6 yields $|\widetilde{A}|\in RV_{-\gamma}$ while $|A|\in RV_{\rho}$ in the first place. Hence, any term associated with $A(t)$ can be neglected, verifying the expansion

$$\frac{U(t)}{U(tx)}\frac{U(tx)-U(t)}{a(t)a_0(t)}=\frac{x^{-\gamma}-1}{-\gamma}+\frac{2}{\gamma}\widetilde{A}(t)\frac{x^{-2\gamma}-1}{-2\gamma}\big(1+o(1)\big),$$

with $a_0(t)=1+2/\gamma A(t)$, since $(1+g(t))^{-1}=1+g(t)+o(g(t))$ for any function g tending to zero as t goes to infinity.

Regarding the case $0<-\rho<\gamma$ or ($0<\gamma<-\rho$ and $l=0$), and because $\widetilde{A}(t)=\gamma/(\gamma+\rho)A(t)$, we can now tidy up (38) as follows:

$$\frac{U(t)}{U(tx)}\frac{U(tx)-U(t)}{a(t)a_0(t)}=\frac{x^{-\gamma}-1}{-\gamma}+\frac{\gamma-\rho}{-\rho(\gamma+\rho)}A(t)\frac{x^{-\gamma+\rho}-1}{-\gamma+\rho}\big(1+o(1)\big),$$

with $(a_0(t))^{-1}=1+(\gamma-\rho)/(\rho(\gamma+\rho))A(t)+o(A(t))$.

Case $\gamma < 0$: We have that $U(\infty)=\lim_{t\to\infty}U(t)\in(0,\infty)$. Bearing this in mind, we shall mimic the steps of the proof for $\gamma>0$ given upstairs. In this sequence, note that through application of Lemma 4(i), Eq. (37) gives rise to

$$\begin{aligned}&\frac{U(t)}{U(tx)}\frac{U(tx)-U(t)}{a(t)}-\frac{x^{\gamma}-1}{\gamma}\\&\quad=\left(\frac{2}{\gamma}\frac{a(t)}{U(\infty)}-\frac{A(t)}{\rho}\right)\frac{x^{\gamma}-1}{\gamma}+\frac{A(t)}{\rho}\frac{x^{\gamma+\rho}-1}{\gamma+\rho}-\frac{2}{\gamma}\frac{a(t)}{U(\infty)}\frac{x^{2\gamma}-1}{2\gamma}+o\big(A(t)\big)+o\big(t^{\gamma}\big).\end{aligned} \tag{39}$$

Whence,

$$\frac{(U(t))^2}{a^*(t)}\left(-\frac{1}{U(tx)}+\frac{1}{U(t)}\right)=\begin{cases}\frac{x^{\gamma}-1}{\gamma}+\frac{A(t)}{\rho}\frac{x^{\gamma+\rho}-1}{\gamma+\rho}+o(A(t)), & \text{if } \gamma<\rho<0,\\ \frac{x^{\gamma}-1}{\gamma}-\frac{2}{\gamma}\frac{a(t)}{U(t)}\frac{x^{2\gamma}-1}{2\gamma}+o(a(t)), & \text{if } \rho<\gamma<0,\end{cases} \tag{40}$$

with $a^{\star}(t)$ as presented in the proposition.

Case $\gamma = 0$: Added the assumption $\lim_{t\to\infty}a(t)=c$ and $c>0$, the limit relation (8) from Lemma 3 entails that

$$\frac{U(tx)-U(t)}{a(t)}=\frac{c}{a(t)}\log x+\frac{f(tx)-f(t)}{a(t)A(t)/\rho}\frac{A(t)}{\rho}=\log x+\frac{A(t)}{\rho}\frac{x^{\rho}-1}{\rho}+o\big(A(t)\big)+o(1). \tag{41}$$

Aside from the latter and supported on (25) displayed in the proof of Theorem 1, we have that

$$\frac{U(tx)}{U(t)}-1=\frac{\log x}{\log t}+o\big((\log t)^{-1}\big),$$

which together with (28) determines the second order condition

$$\lim_{t\to\infty}\frac{\frac{U(t)}{U(tx)}-1}{(\log t)^{-1}}=-\log x,$$

for all $x>0$. Hence, from gathering the above with (41) arises the relation

$$\frac{U(tx)-U(t)}{a(t)}\frac{U(t)}{U(tx)}=\log x+A(t)\frac{1}{\rho}\frac{x^{\rho}-1}{\rho}-\frac{(\log x)^2}{\log t}+o\big(A(t)\big)+o\left(\frac{1}{\log t}\right), \tag{42}$$

or equivalently,

$$\frac{(U(t))^2}{a(t)}\left(-\frac{1}{U(tx)}+\frac{1}{U(t)}\right)=\log x-\frac{1}{\log t}\left((\log x)^2-A(t)\log t\frac{1}{\rho}\frac{x^{\rho}-1}{\rho}+o\big(A(t)\log t\big)+o(1)\right).$$

As before, the result follows by noticing that $|A(t)\log t|\in RV_{\rho}$, $\rho<0$ and $a(t)/U(t)\sim(\log t)^{-1}$, $t\to\infty$. □

Remark 10. Provided the slightly more restrictive condition that $A(t)\sim c_2^{\star}t^{\rho}$, as $t\to\infty$, also assumed in (9), the case $\gamma=\rho$ would follow from (39) in the similar trivial manner as it yields (40).

Lemma 11. *Let $U \in 2ERV(\gamma, 0)$, $\gamma \neq 0$, and assume $U(\infty) \in (0, \infty]$. Then,*

$$\lim_{t\to\infty} \frac{\frac{(U(t))^2}{a^\star(t)}\left(-\frac{1}{U(tx)}+\frac{1}{U(t)}\right)-\frac{x^{-|\gamma|}-1}{-|\gamma|}}{A^\star(t)} = K^\star_{\gamma,0}(x) := \frac{x^{-|\gamma|}}{\gamma}\log x, \tag{43}$$

for all $x > 0$, where $A^\star = A$ and $a^\star = 1 - \gamma^{-1}A$.

Proof. The proof is virtually sustained on (i) and (ii) of Lemma 3. Note that, with a positive function f defined as $f(t) = U(t)$ if $\gamma > 0$ and $f(t) = U(\infty) - U(t)$ if $\gamma < 0$, we have

$$\frac{U(t)}{a(t)}\left(\frac{U(tx)}{U(t)}-1\right)-\frac{x^\gamma-1}{\gamma} = \gamma\left(\frac{f(t)}{a(t)}-\frac{1}{\gamma}\right)\frac{x^\gamma-1}{\gamma}+\frac{f(t)}{a(t)}\left(\frac{f(tx)}{f(t)}-x^\gamma\right). \tag{44}$$

Now, since (7) in Lemma 3 can be rephrased as a second order condition, more concretely,

$$\lim_{t\to\infty}\frac{\frac{f(tx)}{f(t)}-x^\gamma}{A(t)} = x^\gamma \log x, \quad \text{all } x>0,$$

and the equivalence between (4) and (8) of Lemma 3(i) (also embraced in Lemma 3(ii)) yields

$$\lim_{t\to\infty}\frac{\frac{f(t)}{a(t)}-\frac{1}{\gamma}}{A(t)} = -\frac{1}{\gamma^2}, \tag{45}$$

we only have to apply the device (28) in order to make progression from (44) to the stated result for $\gamma < 0$. Regarding $\gamma > 0$, combination of Proposition 6 with (44) allows to obtain:

$$\frac{U(t)}{U(tx)}\frac{U(tx)-U(t)}{a(t)} = \left(1-\frac{A(t)}{\gamma}\right)\frac{x^{-\gamma}-1}{-\gamma}+\frac{A(t)}{\gamma}x^{-\gamma}\log x + o\big(A(t)\big),$$

as $t \to \infty$, from which the result follows readily.

In the same spirit of Corollary 7, we can easily see that (33) (and also (43)) renders a second order refinement of (30), in the case $\gamma \leqslant 0$. This yields from the fact that $0 \leqslant -y - \log(1-y) \leqslant y^2/(2(1-y))$ with $y := 1 - U(t)/U(tx)$, all $x \geqslant 1$, and can be verified by applying uniform inequalities (31) and (32). In particular, observe that for any $\varepsilon > 0$, we have $(U(tx)/U(t)-1)^2 \leqslant 2\{(\widetilde{A}(t)K_{\gamma,\widetilde{\rho}}(x))^2 + (\varepsilon x^{\widetilde{\rho}+\varepsilon})^2\}$, uniformly for each $x \geqslant 1$. Now, note that $|\widetilde{A}| \in RV_{\widetilde{\rho}}$ and thus take t_0 such that $\varepsilon t^{-\widetilde{\rho}-\varepsilon} < 1$ for $t \geqslant t_0$. Whence, $(U(tx)/U(t)-1)^2 \leqslant 2(\widetilde{A}(t))^2\{K^2_{\gamma,\widetilde{\rho}}(x) + x^{2\widetilde{\rho}+\varepsilon}\}$ while $U(tx)/U(t) < (1+\varepsilon)x^{\gamma_+ + \varepsilon}$. Similar uniform bounds to those of Proposition 8 are also available for $0 < x < 1$ (see e.g. [4,6]) meaning that the scope of this reasoning can be lengthened to every $x > 0$. Therefore, combining the result displayed in Corollary 7 for $\gamma > 0$ with Proposition 9 and Lemma 11 for $\gamma \leqslant 0$, we can retrieve the useful and interesting second order relation for $\log U$ given in Lemma B.3.16 of [4]. Hence, the following lemma trails along with Proposition 9:

Lemma 12. *(See [4, Lemma B.3.16].) Under conditions of Proposition 9,*

$$\lim_{t\to\infty}\frac{\frac{\log U(tx)-\log U(t)}{a^\star(t)/U(t)}-\frac{x^{\gamma_-}-1}{\gamma_-}}{B(t)} = \begin{cases} \frac{x^{\gamma_-+\rho^\star}-1}{\gamma_-+\rho^\star}, & \gamma \neq 0, \\ \frac{(\log x)^2}{2}, & \gamma = 0, \end{cases}$$

for all $x \geqslant 0$, where $|B(t)| \in RV_{\rho^\star}$,

$$B(t) = \begin{cases} \frac{1}{\gamma}\widetilde{A}(t), & \gamma > 0, \\ \frac{1}{\gamma}\frac{a(t)}{U(t)}, & \rho < \gamma < 0, \\ \frac{1}{\rho}A(t), & \gamma < \rho < 0, \\ \frac{a(t)}{U(t)}, & \rho < \gamma = 0. \end{cases}$$

Remark 13. Owing to the limit (45) obtained in the proof of Lemma 11, the limit above vanishes whenever $\gamma > 0 = \rho$. The latter yields accordance with [4, p. 103].

Remark 14. Bearing on Remark 5, it is not always true that the second order extended regular variation of $\log U$ implies (4).

The following theorem arises as a third order refinement of the second order statement in Proposition 6 (and ulterior counterpart in the sense of Theorem 1). Because when $\gamma + \rho > 0$ and $\rho < 0$, we have at the expenses of (10) and Lemma 2(ii) that $A(t) \sim (a(t)/U(t) - \gamma)(\gamma + \rho)/\gamma$, as $t \to \infty$, then in view of (28), Theorem 15 below is only relevant when $\gamma + \rho < 0$. On the same account, the case of $0 < \gamma < -\rho$ and $l = 0$ may also be excluded.

Theorem 15. *Assume that* (4) *holds for* $\rho < 0$ *with the restriction* $\gamma + \rho \leqslant 0$. *Then*

$$\lim_{t\to\infty} \frac{\frac{\frac{U(t)}{U(tx)} - x^{-\gamma_+}}{a(t)/U(t) - \gamma_+} + x^{\gamma}\frac{x^{-\gamma}-1}{-\gamma}}{\bar{A}(t)} = \begin{cases} -x^{-\gamma}\frac{x^{\rho}-1}{\rho}, & 0 < -\rho/2 < \gamma \text{ or } (0 < \gamma \leqslant -\rho/2 \text{ and } l = 0), \\ -x^{-\gamma_+}\frac{x^{-2|\gamma|}-1}{-2|\gamma|}, & (0 < \gamma < -\rho/2 \text{ and } l \neq 0) \text{ or } \gamma = -\rho \text{ or } \gamma \leqslant 0, \end{cases} \tag{46}$$

for all $x > 0$, *where*

$$\bar{A}(t) = \begin{cases} \frac{A(t)}{a(t)/U(t) - \gamma_+}, & 0 < -\rho/2 < \gamma \text{ or } (0 < \gamma \leqslant -\rho/2 \text{ and } l = 0), \\ -\frac{2}{|\gamma|}\left(\frac{a(t)}{U(t)} - \gamma_+\right), & (0 < \gamma < -\rho/2 \text{ and } l \neq 0) \text{ or } \gamma = -\rho \text{ or } \gamma \leqslant 0, \end{cases}$$

with l standing for the limit defined in Lemma 4(ii).

Proof. In order to have a grasp of the third order term, it is imperative to expand (28) further and thereby obtain, as $t \to \infty$,

$$\frac{U(t)}{U(tx)} - x^{-\gamma_+} = -x^{-2\gamma_+}\left(\frac{U(tx)}{U(t)} - x^{\gamma_+}\right) + 2x^{-3\gamma_+}\left(\frac{U(tx)}{U(t)} - x^{\gamma_+}\right)^2\big(1 + o(1)\big). \tag{47}$$

If $0 < \gamma < -\rho$ and $l \neq 0$, we get from (20) in a similar way as we have obtained (21), that

$$\begin{aligned} \frac{U(t)}{U(tx)} - x^{-\gamma} = {} & -x^{-\gamma}\frac{x^{\rho}-1}{\rho}\frac{\gamma}{\gamma+\rho}A(t) - x^{-\gamma}\frac{x^{-\gamma}-1}{-\gamma}\left(\frac{a(t)}{U(t)} - \gamma\right) \\ & - x^{-\gamma}\frac{x^{-2\gamma}-1}{-2\gamma}\frac{2}{\gamma}\left(\frac{a(t)}{U(t)} - \gamma\right)^2 + o\big(A(t)\big) + o\left(\left(\frac{a(t)}{U(t)} - \gamma\right)^2\right). \end{aligned}$$

Bearing on (11) from Lemma 4, the result for positive γ follows immediately.

If $\gamma = -\rho$, we use similar calculations as before upon (17). The result is attained by noting that Theorem 1 ensures $(a(t)/U(t) - \gamma)$ as of regular variation with index $-\gamma$, hence $(a(t)/U(t) - \gamma)^2/|A(t)|$ also belongs to $RV_{-\gamma}$ which means that it becomes negligible.

If $\gamma < 0$, the result follows through replacement in the expansion (23): by the one hand, Lemma 4(i) states that $ct^{\gamma}/a(t) - 1 = -\rho^{-1}A(t)(1 + o(1))$. On the other hand, we have from Lemma 2(ii) in conjunction with (8) pertaining to $\gamma + \rho < 0$, that

$$\frac{a(t)A(t)}{\rho(f(\infty) - f(t))} \underset{t\to\infty}{\longrightarrow} -(\gamma + \rho),$$

whence

$$\frac{f(\infty) - f(t)}{ct^{\gamma}} = -\frac{A(t)}{\rho(\gamma+\rho)} + \frac{A^2(t)}{\gamma+\rho} + o\big(A^2(t)\big).$$

The precise statement is therefore attained by straightforward calculations upon (23).

Finally, in the case of $\gamma = 0$, the fact that $|A(t)|(\log t)^2 \in RV_{\rho}$ and $a(t)/U(t) \sim (\log t)^{-1}$ yields the result (see text just before relation (25)). □

5. Example apropos to the asymptotics for γ-estimators: the maximum likelihood estimator

While restricting attention to a top portion of the original sample, the Generalized Pareto distribution $F_{\gamma}(w) = 1 - (1 + \gamma w)^{-1/\gamma}$, $w > 0$ if $\gamma \geqslant 0$ and $0 < w < -1/\gamma$ if $\gamma < 0$, shall come into play since it stems from the fundamental results of [1] and [10] as the limiting distribution for the excesses $W_i = X_i - u \mid X_i > u$, $i = 1, \ldots, k_u$ over a sufficiently high threshold u. The maximum likelihood approach may then be applied under the assumption that the k_u excesses above the threshold $u > 0$ follow exactly a GP distribution, provided the scale normalization a_u, which accommodates the influence of the designated threshold. Hence, we are actually making use of a basic procedure for parametric estimation via replacement of a limit by an exact equality. In order to construct confidence statements for the inference, we need to rely on that $\sqrt{k}(\hat{\gamma}_n - \gamma)$ attains the normal limit, as $n \to \infty$, under extra mild conditions. A convenient way to assure this is by assuming that $U \in 2ERV(\gamma, \rho)$. We will see that, in the heavy tailed case, our approach to the asymptotic normality for the Maximum Likelihood estimator $\hat{\gamma}$ assigns motivation to Theorem 15. We shall present a mitigated version of the proof by Drees et al. [7] (see also [4]) for the heavy tailed case, where we wish to keep the asymptotic bias under scrutiny. Moreover, we do not make direct use of the tail quantile process here.

Let $X_1, X_2, \ldots, X_n$ be independent random variables with the same distribution function F and let $X_{1,n} \leqslant X_{2,n} \leqslant \cdots \leqslant X_{n,n}$ denote their ascending order statistics. Then $X_{n-k,n}$ denotes the $(k+1)$th largest order statistics which we shall designate as the eventually high (random) threshold: $u = X_{n-k,n}$. In the absence of a closed form estimator, the asymptotic normality of $\hat{\gamma}$ follows from the likelihood equations

$$\begin{cases} \dfrac{1}{k}\displaystyle\sum_{i=0}^{k-1} \log\left(1+\frac{\hat{\gamma}}{\hat{a}(n/k)}(X_{n-i,n}-X_{n-k,n})\right)=\hat{\gamma}, \\ \dfrac{1}{k}\displaystyle\sum_{i=0}^{k-1} \left(1+\frac{\hat{\gamma}}{\hat{a}(n/k)}(X_{n-i,n}-X_{n-k,n})\right)^{-1}=\dfrac{1}{1+\hat{\gamma}}, \end{cases} \tag{48}$$

where $\hat{a}$ denotes an estimator for the scale function a. Note that the leading term on both equations can be rewritten as

$$\frac{1}{1+\frac{\hat{\gamma}}{\hat{a}(n/k)}(X_{n-i,n}-X_{n-k,n})}=\frac{X_{n-k,n}/X_{n-i,n}}{1+(\hat{\gamma}\frac{X_{n-k,n}}{\hat{a}(n/k)}-1)(1-\frac{X_{n-k,n}}{X_{n-i,n}})}, \tag{49}$$

for every $i=0,1,\dots,k-1$. Now, apply the equality

$$\frac{1}{1+x}=1-x+\frac{x^2}{1+x}, \quad x\neq -1$$

by assigning $x:=(\hat{\gamma}X_{n-k,n}/\hat{a}(n/k)-1)(1-X_{n-k,n}/X_{n-i,n})$, we thus have

$$\begin{aligned} \frac{1}{1+\frac{\hat{\gamma}}{\hat{a}(n/k)}(X_{n-i,n}-X_{n-k,n})} &= \frac{X_{n-k,n}}{X_{n-i,n}}-\left(\hat{\gamma}\frac{X_{n-k,n}}{\hat{a}(n/k)}-1\right)\left(1-\frac{X_{n-k,n}}{X_{n-i,n}}\right)\frac{X_{n-k,n}}{X_{n-i,n}} \\ &\quad+\left(\hat{\gamma}\frac{X_{n-k,n}}{\hat{a}(n/k)}-1\right)^2\frac{(1-\frac{X_{n-k,n}}{X_{n-i,n}})^2\frac{X_{n-k,n}}{X_{n-i,n}}}{1+(\hat{\gamma}\frac{X_{n-k,n}}{\hat{a}(n/k)}-1)(1-\frac{X_{n-k,n}}{X_{n-i,n}})}. \end{aligned}$$

Since $0<X_{n-k,n}/X_{n-i,n}<1$, with probability one, for every $i=0,1,\dots,k-1$, if $\hat{\gamma}$ and $\hat{a}(n/k)$ are consistent estimators of γ and $a(n/k)$, respectively, then

$$\frac{(1-\frac{X_{n-k,n}}{X_{n-i,n}})^2\frac{X_{n-k,n}}{X_{n-i,n}}}{1+(\hat{\gamma}\frac{X_{n-k,n}}{\hat{a}(n/k)}-1)(1-\frac{X_{n-k,n}}{X_{n-i,n}})}=\left(1-\frac{X_{n-k,n}}{X_{n-i,n}}\right)^2\frac{X_{n-k,n}}{X_{n-i,n}}\big(1+o_p(1)\big),$$

as $n\to\infty$, uniformly in $i=0,1,\dots,k-1$. Hence,

$$\begin{aligned} \frac{1}{k}\sum_{i=0}^{k-1}\frac{1}{1+\frac{\hat{\gamma}}{\hat{a}(n/k)}(X_{n-i,n}-X_{n-k,n})} &= \frac{1}{k}\sum_{i=0}^{k-1}\frac{X_{n-k,n}}{X_{n-i,n}}-\left(\hat{\gamma}\frac{a(n/k)}{\hat{a}(n/k)}-\frac{a(n/k)}{X_{n-k,n}}\right)\frac{1}{k}\sum_{i=0}^{k-1}\left(\frac{X_{n-k,n}}{X_{n-i,n}}\right)^2\frac{X_{n-i,n}-X_{n-k,n}}{a(n/k)} \\ &\quad+\left(\hat{\gamma}\frac{a(n/k)}{\hat{a}(n/k)}-\frac{a(n/k)}{X_{n-k,n}}\right)^2\big(1+o_p(1)\big)\frac{1}{k}\sum_{i=0}^{k-1}\left(\frac{X_{n-k,n}}{X_{n-i,n}}\right)^3\left(\frac{X_{n-i,n}-X_{n-k,n}}{a(n/k)}\right)^2. \end{aligned}$$

Now, given a random variable X with distribution function F and quantile type function U, we have the equality in distribution $X\stackrel{d}{=}U(Y)$ with Y a standard Pareto random variable with distribution function $1-(y\vee 1)^{-1}$. Naturally, if we consider a sample $\{X_i\}_{i=1}^n$ of independent and identically distributed random variables from the same parent F and let $X_{1,n}\leqslant X_{2,n}\leqslant\cdots\leqslant X_{n,n}$ be their ascending order statistics, we get $\{X_{n-i,n}\}_{i=0}^{n-1}\stackrel{d}{=}\{U(Y_{n-i,n})\}_{i=0}^{n-1}$. In order to apply (26) and appropriate uniform inequalities in view of (32), bear in mind that if $k=k_n$ is an intermediate sequence, i.e., $k\to\infty$ and $k/n\to 0$ as $n\to\infty$, then $(k/n)Y_{n-k,n}\xrightarrow[n\to\infty]{P}1$ which entails in turn that $Y_{n-k,n}\xrightarrow[n\to\infty]{P}\infty$. Hence, if $\gamma>0$,

$$\frac{1}{k}\sum_{i=0}^{k-1}\frac{X_{n-k,n}}{X_{n-i,n}}\stackrel{d}{=}\frac{1}{k}\sum_{i=0}^{k-1}\frac{U(Y_{n-k,n})}{U(\frac{Y_{n-i,n}}{Y_{n-k,n}}Y_{n-k,n})}=\frac{1}{k}\sum_{i=0}^{k-1}\left(\frac{Y_{n-i,n}}{Y_{n-k,n}}\right)^{-\gamma}+\widetilde{A}(Y_{n-k,n})\frac{1}{k}\sum_{i=0}^{k-1}K_{\gamma,\widetilde{\rho}}\left(\frac{Y_{n-i,n}}{Y_{n-k,n}}\right)+o_p\left(\widetilde{A}\left(\frac{n}{k}\right)\right). \tag{50}$$

By application of Rényi's representation (cf. Lemma 3.4.1 of [4]), we see that $\{Y_{n-i+1,n}/Y_{n-k,n}\}_{i=1}^k\stackrel{d}{=}\{Y^\star_{i,k}\}_{i=1}^k$, where $Y_1^\star,Y_2^\star,\dots,Y_k^\star$ are independent standard Pareto random variables distribution function $1-(y\vee 1)^{-1}$. Therefore, with any measurable function g, we have the equality in distribution

$$\frac{1}{k}\sum_{i=0}^{k-1}g(Y_{n-i,n}/Y_{n-k,n})\stackrel{d}{=}\frac{1}{k}\sum_{i=1}^{k}g\big(Y_i^\star\big). \tag{51}$$

Hence, the right-hand side of (50) rephrases as

$$\frac{1}{k}\sum_{i=1}^{k}\big(Y_i^\star\big)^{-\gamma}+\widetilde{A}\left(\frac{n}{k}\right)\frac{1}{k}\sum_{i=1}^{k}K_{\gamma,\widetilde{\rho}}\big(Y_i^\star\big)+o_p\left(\widetilde{A}\left(\frac{n}{k}\right)\right)=\frac{1}{1+\gamma}+\frac{P_1}{\sqrt{k}}+\widetilde{A}\left(\frac{n}{k}\right)d_{\gamma,\widetilde{\rho}}+o_p\left(\frac{1}{\sqrt{k}}\right)+o_p\left(\widetilde{A}\left(\frac{n}{k}\right)\right),$$

where

$$d_{\gamma,\widetilde{\rho}} = \int_1^\infty K_{\gamma,\widetilde{\rho}}(y)\frac{dy}{y^2},$$

$$\sqrt{k}\left(\frac{1}{k}\sum_{i=1}^{k}\left(Y_i^\star\right)^{-\gamma} - \frac{1}{1+\gamma}\right) \xrightarrow[k\to\infty]{d} P_1 \sim N\left(0,\sigma_1^2\right).$$

With respect to the other important random term featuring in the likelihood equations (48), we have that

$$\frac{a(n/k)}{X_{n-k,n}} = \frac{a(n/k)}{U(n/k)}\left(\left(\frac{k}{n}Y_{n-k,n}\right)^{-\gamma} + \widetilde{A}\left(\frac{n}{k}\right)K_{\gamma,\rho}\left(\frac{k}{n}Y_{n-k,n}\right) + o_p(1)\widetilde{A}\left(\frac{n}{k}\right)\right)$$
$$= \gamma + \frac{B}{\sqrt{k}} + o_p\left(\frac{1}{\sqrt{k}}\right) + o_p\left(\widetilde{A}\left(\frac{n}{k}\right)\right).$$

Now, combining (4) with (26), we get

$$\frac{1}{k}\sum_{i=0}^{k-1}\left(\frac{X_{n-k,n}}{X_{n-i,n}}\right)^2\frac{X_{n-i,n}-X_{n-k,n}}{a(n/k)} \xrightarrow[n\to\infty]{P} \int_1^\infty \frac{y^{-\gamma}-y^{-2\gamma}}{\gamma}\frac{dy}{y^2} = \frac{1}{(1+\gamma)(1+2\gamma)}.$$

Thus, under the same assumption as in [7] that $|\hat{\gamma}a(n/k)/\hat{a}(n/k) - \gamma| = O_p(k^{-1/2})$,

$$\frac{1}{k}\sum_{i=0}^{k-1}\frac{1}{1+\frac{\hat{\gamma}}{\hat{a}(n/k)}(X_{n-i,n}-X_{n-k,n})} = \frac{1}{1+\gamma} + \frac{P_1}{\sqrt{k}} + \frac{1}{(1+\gamma)(1+2\gamma)}\frac{B}{\sqrt{k}} + \widetilde{A}\left(\frac{n}{k}\right)d_{\gamma,\widetilde{\rho}}$$
$$- \left(\hat{\gamma}\frac{\hat{a}(n/k)}{a(n/k)} - \gamma\right)\frac{1}{(1+\gamma)(1+2\gamma)} + o_p\left(\frac{1}{\sqrt{k}}\right) + o_p\left(\widetilde{A}\left(\frac{n}{k}\right)\right).$$

The standard normal random variable B is independent of P_1.

Concerning the logarithm of (49) for each $i = 0, 1, \ldots, k-1$,

$$-\log\left(1+\frac{\hat{\gamma}}{\hat{a}(n/k)}(X_{n-i,n}-X_{n-k,n})\right) = \log\left(\frac{X_{n-k,n}}{X_{n-i,n}}\right) - \log\left(1+\left(\hat{\gamma}\frac{X_{n-k,n}}{\hat{a}(n/k)}-1\right)\left(1-\frac{X_{n-k,n}}{X_{n-i,n}}\right)\right),$$

our interest lays upon

$$-\log\left(1+\frac{\hat{\gamma}}{\hat{a}(n/k)}(X_{n-i,n}-X_{n-k,n})\right) - \log\left(\frac{X_{n-k,n}}{X_{n-i,n}}\right) + \left(\hat{\gamma}\frac{X_{n-k,n}}{\hat{a}(n/k)}-1\right)\left(1-\frac{X_{n-k,n}}{X_{n-i,n}}\right)$$
$$= \left(\hat{\gamma}\frac{X_{n-k,n}}{\hat{a}(n/k)}-1\right)\left(1-\frac{X_{n-k,n}}{X_{n-i,n}}\right) - \log\left(\frac{X_{n-k,n}}{X_{n-i,n}}\right) + \left(\hat{\gamma}\frac{X_{n-k,n}}{\hat{a}(n/k)}-1\right)\left(1-\frac{X_{n-k,n}}{X_{n-i,n}}\right). \tag{52}$$

Given that $\hat{\gamma}$ and $\hat{a}(n/k)$ are consistent estimators of γ and $a(n/k)$, respectively, we can use the well-known inequality

$$0 \leqslant x - \log(1+x) \leqslant \frac{x^2}{2(1\wedge(1+x))}, \quad x > -1,$$

with x replaced by $(\hat{\gamma}X_{n-k,n}/\hat{a}(n/k) - 1)(1 - X_{n-k,n}/X_{n-i,n})$ as before, to conclude with respect to the right-hand side of (52) that

$$\left(\hat{\gamma}\frac{X_{n-k,n}}{\hat{a}(n/k)}-1\right)\left(1-\frac{X_{n-k,n}}{X_{n-i,n}}\right) - \log\left(1+\left(\hat{\gamma}\frac{X_{n-k,n}}{\hat{a}(n/k)}-1\right)\left(1-\frac{X_{n-k,n}}{X_{n-i,n}}\right)\right)$$
$$\leqslant \frac{(\hat{\gamma}\frac{X_{n-k,n}}{\hat{a}(n/k)}-1)^2(1-\frac{X_{n-k,n}}{X_{n-i,n}})^2}{2\min(1, 1+(\hat{\gamma}\frac{X_{n-k,n}}{\hat{a}(n/k)}-1)(1-\frac{X_{n-k,n}}{X_{n-i,n}}))} = \frac{1}{2}\left(\hat{\gamma}\frac{X_{n-k,n}}{\hat{a}(n/k)}-1\right)^2\left(1-\frac{X_{n-k,n}}{X_{n-i,n}}\right)^2\left(1+o_p(1)\right),$$

uniformly in $i = 0, 1, \ldots, k-1$. Hence, we can write

$$\frac{1}{k}\sum_{i=0}^{k-1} -\log\left(1+\frac{\hat{\gamma}}{\hat{a}(n/k)}(X_{n-i,n}-X_{n-k,n})\right) - \frac{1}{k}\sum_{i=0}^{k-1}\log\frac{X_{n-k,n}}{X_{n-i,n}} + \left(\hat{\gamma}\frac{X_{n-k,n}}{\hat{a}(n/k)}-1\right)\frac{1}{k}\sum_{i=0}^{k-1}\left(1-\frac{X_{n-k,n}}{X_{n-i,n}}\right) \leqslant R_n(k),$$

where

$$R_n(k) := \frac{1}{2}\left(\hat{\gamma}\,\frac{X_{n-k,n}}{\hat{a}(n/k)} - 1\right)^2 \bigl(1 + o_p(1)\bigr)\frac{1}{k}\sum_{i=0}^{k-1}\left(1 - \frac{X_{n-k,n}}{X_{n-i,n}}\right)^2.$$

Again, assuming that $|\hat{\gamma}a(n/k)/\hat{a}(n/k) - \gamma| = O_p(k^{-1/2})$ holds, $R_n(k) = o_p(k^{-1/2})$ and therefore

$$\begin{aligned}
&\frac{1}{k}\sum_{i=0}^{k-1} -\log\left(1 + \frac{\hat{\gamma}}{\hat{a}(n/k)}(X_{n-i,n} - X_{n-k,n})\right)\\
&\quad = \frac{1}{k}\sum_{i=0}^{k-1}\log\frac{X_{n-k,n}}{X_{n-i,n}} - \left(\hat{\gamma}\,\frac{X_{n-k,n}}{\hat{a}(n/k)} - 1\right)\frac{1}{k}\sum_{i=0}^{k-1}\left(1 - \frac{X_{n-k,n}}{X_{n-i,n}}\right) + o_p\left(\frac{1}{\sqrt{k}}\right)\\
&\quad = \frac{1}{k}\sum_{i=0}^{k-1}\log\frac{X_{n-k,n}}{X_{n-i,n}} - \left(\hat{\gamma}\,\frac{a(n/k)}{\hat{a}(n/k)} - \frac{a(n/k)}{X_{n-k,n}}\right)\frac{1}{k}\sum_{i=0}^{k-1}\frac{X_{n-k,n}}{X_{n-i,n}}\,\frac{X_{n-i,n} - X_{n-k,n}}{a(n/k)} + o_p\left(\frac{1}{\sqrt{k}}\right).
\end{aligned}$$

Now, from (4) together with (26) we have, for an intermediate sequence $k = k_n$,

$$\frac{1}{k}\sum_{i=0}^{k-1}\frac{X_{n-k,n}}{X_{n-i,n}}\,\frac{X_{n-i,n} - X_{n-k,n}}{a(n/k)} \xrightarrow[n\to\infty]{P} \int_1^\infty \frac{y^{-\gamma} - 1}{-\gamma}\,\frac{dy}{y^2} = \frac{1}{1+\gamma}.$$

Similarly as before, we have from (29) that

$$\begin{aligned}
\frac{1}{k}\sum_{i=0}^{k-1} -\log\left(1 + \frac{\hat{\gamma}}{\hat{a}(n/k)}(X_{n-i,n} - X_{n-k,n})\right) &= -\gamma - \frac{P_0}{\sqrt{k}} + \frac{1}{1+\gamma}\,\frac{B}{\sqrt{k}} - \widetilde{A}\left(\frac{n}{k}\right)\widetilde{d}_{\gamma,\rho} - \left(\frac{a(n/k)}{\hat{a}(n/k)} - \gamma\right)\frac{1}{1+\gamma}\\
&\quad + o_p\left(\frac{1}{\sqrt{k}}\right) + o_p\left(\widetilde{A}\left(\frac{n}{k}\right)\right),
\end{aligned}$$

where

$$\widetilde{d}_{\gamma,\rho} = \int_1^\infty \widetilde{K}_{\gamma,\rho}(y)\,\frac{dy}{y^2},$$

$$\sqrt{k}\left(\frac{1}{k}\sum_{i=1}^{k}\log\bigl(Y_i^\star\bigr) - \gamma\right)\xrightarrow[k\to\infty]{d} P_0 \sim N\bigl(0, \sigma_0^2\bigr),$$

and the standard normal random variable B is independent of P_0.

Therefore, the system of Eqs. (48) is equivalent to

$$\sqrt{k}(\hat{\gamma} - \gamma) = P_0 - \frac{1}{1+\gamma}B + \sqrt{k}\left(\hat{\gamma}\,\frac{a(n/k)}{\hat{a}(n/k)} - \gamma\right)\frac{1}{1+\gamma} + \sqrt{k}\widetilde{A}\left(\frac{n}{k}\right)\widetilde{d}_{\gamma,\rho} + o_p\left(\sqrt{k}\widetilde{A}\left(\frac{n}{k}\right)\right) + o_p(1), \tag{53}$$

$$\begin{aligned}
\sqrt{k}\left(\frac{1}{1+\hat{\gamma}} - \frac{1}{1+\gamma}\right) &= P_1 + \frac{1}{(1+\gamma)(1+2\gamma)}B - \sqrt{k}\left(\hat{\gamma}\,\frac{a(n/k)}{\hat{a}(n/k)} - \gamma\right)\frac{1}{(1+\gamma)(1+2\gamma)}\\
&\quad + \sqrt{k}\widetilde{A}\left(\frac{n}{k}\right)d_{\gamma,\rho} + o_p\left(\sqrt{k}\widetilde{A}\left(\frac{n}{k}\right)\right) + o_p(1),
\end{aligned} \tag{54}$$

where

$$\widetilde{d}_{\gamma,\rho} = \begin{cases} \frac{\gamma}{(\gamma+\rho)(1-\rho)}, & 0 \leqslant -\rho < \gamma \text{ or } (0 < \gamma < -\rho \text{ and } l = 0),\\ -\frac{\gamma^2}{1+\gamma}, & (0 < \gamma < -\rho \text{ and } l \neq 0) \text{ or } \gamma = -\rho, \end{cases}$$

$$d_{\gamma,\rho} = \begin{cases} -\frac{\gamma}{(1+\gamma)(\gamma+\rho)(1+\gamma-\rho)}, & 0 \leqslant -\rho < \gamma \text{ or } (0 < \gamma < -\rho \text{ and } l = 0),\\ \frac{\gamma^2}{(1+\gamma)(1+2\gamma)}, & (0 < \gamma < -\rho \text{ and } l \neq 0) \text{ or } \gamma = -\rho, \end{cases}$$

and (P_0, P_1) is a normally distributed random vector, with mean zero and covariance matrix

$$\begin{bmatrix} \gamma^2 & -\frac{\gamma^2}{(1+\gamma)^2}\\ -\frac{\gamma^2}{(1+\gamma)^2} & \frac{\gamma^2}{(1+\gamma)^2(1+2\gamma)} \end{bmatrix},$$

independent of the standard normal random variable B. Now, Eq. (53) and the main assumption $|\hat{\gamma}a(n/k)/\hat{a}(n/k) - \gamma| = O_p(k^{-1/2})$ yield that $|\hat{\gamma} - \gamma| = O_p(k^{-1/2})$. Hence,

$$\sqrt{k}\left(\frac{1}{1+\hat{\gamma}} - \frac{1}{1+\gamma}\right) = -\sqrt{k}\frac{\hat{\gamma}-\gamma}{(\hat{\gamma}-\gamma)(1+\gamma)+(1+\gamma)^2} = -\sqrt{k}\frac{\hat{\gamma}-\gamma}{(1+\gamma)^2} + o_p(1). \tag{55}$$

Apart from the latter, note that

$$\sqrt{k}\left(\hat{\gamma}\frac{a(n/k)}{\hat{a}(n/k)} - \gamma\right) = \sqrt{k}\frac{a(n/k)}{\hat{a}(n/k)}\left(\hat{\gamma} - \gamma - \gamma\left(\frac{\hat{a}(n/k)}{a(n/k)} - 1\right)\right) = \sqrt{k}(\hat{\gamma}-\gamma) - \gamma\sqrt{k}\left(\frac{\hat{a}(n/k)}{a(n/k)} - 1\right) + o_p(1).$$

Using the above upon (54) together with (55), we finally get

$$\sqrt{k}(\hat{\gamma}-\gamma) = \left(\frac{\gamma}{1+\gamma}\right)^2\big(P_0 + (1+2\gamma)P_1\big) + \sqrt{k}\widetilde{A}\left(\frac{n}{k}\right)\big(\widetilde{d}_{\gamma,\rho} + (1+2\gamma)d_{\gamma,\rho}\big) + o_p\left(\sqrt{k}\widetilde{A}\left(\frac{n}{k}\right)\right) + o_p(1),$$

$$\begin{aligned}\sqrt{k}\left(\frac{\hat{a}(n/k)}{a(n/k)} - 1\right) = {} & \frac{1+\gamma}{\gamma}P_0 - \left(\frac{1+\gamma}{\gamma}\right)^2 P_1 + \frac{1}{\gamma}B + \sqrt{k}\widetilde{A}\left(\frac{n}{k}\right)\left(\frac{1+\gamma}{\gamma}\left(1 - \frac{1+\gamma}{\gamma}\right)\widetilde{d}_{\gamma,\rho} - \frac{(1+\gamma)^2(1+2\gamma)}{\gamma^2}d_{\gamma,\rho}\right) \\ & + o_p\left(\sqrt{k}\widetilde{A}\left(\frac{n}{k}\right)\right) + o_p(1).\end{aligned}$$

In the particular case of ($0 < \gamma < -\rho$ and $l \neq 0$) or $\gamma = -\rho$, where $\widetilde{A}(t) = a(t)/U(t) - \gamma_+$, we find that the bias term $\widetilde{d}_{\gamma,\rho} + (1+2\gamma)d_{\gamma,\rho}$ is equal to zero which makes it imperative to call on the subsequent term in the approximation. Bearing on Theorem 15, we obtain the following expansion by similar calculations as before:

$$\sqrt{k}(\hat{\gamma}-\gamma) = \left(\frac{\gamma}{1+\gamma}\right)^2\big(P_0 + (1+2\gamma)P_1\big) + \sqrt{k}A\left(\frac{n}{k}\right)\frac{1+\gamma}{(1-\rho)(1+\gamma-\rho)} + o_p\left(\sqrt{k}A\left(\frac{n}{k}\right)\right) + o_p(1),$$

valid for every $\gamma > 0$. This is the same bias obtained by [7]. We have now acknowledged this dominant component of the bias as a sometimes third order asymptotic bias.

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